A. $f : \mathbb{R}^2 \mapsto \mathbb{R}$ $\quad \mathcal{C}^1(\mathbb{R}^2)$ t.c. $f(x,y) = x^2 - y^2 + o(x^2+y^2)$

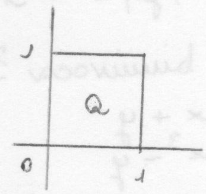
il Polinomio di Taylor di f coincide con f.

$$\Phi_{f,2}\left(0, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{1}{2}(2x^2 - 2y^2)$$

$$Hf\big|_0 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow 0 \text{ pto di sella}$$

B. $\omega(x,y) = e^{x^2} dx + \arctan(xy)\, dy = M dx + N dy$

Q quadrato di vertici $(0,0)$ $(1,0)$, $(1,1)$, $(0,1)$



$$\int_{\partial Q^+} \omega(x,y) = \int_Q N_x - M_y \, dx dy$$

$$\int_Q \left[\frac{d}{dx}(\arctan(xy)) - \frac{d}{dy}\left(e^{x^2}\right) \right]$$

$$= \int_Q \left[\frac{d}{dx}(\arctan(xy)) \right]$$

$$= \int_0^1 dy \int_0^1 dx \, \frac{d}{dx}(\arctan(xy))$$

$$= \int_0^1 dy \, \arctan(xy)\Big|_0^1$$

$$= \int_0^1 dy \, \arctan(y)$$

$$= y \arctan y \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2y}{1+y^2} dy$$

$$= \arctan(1) - \frac{1}{2}\ln(1+x^2)\Big|_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

THANKS, FLEUR-DE-LIS!

C. $x' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ La matrice non è evidentemente diagonalizzabile:

$\det(A - \lambda I) = (1-\lambda)^2 = 0$

$\lambda = 1 \Rightarrow m_a(1) = 2$

$V_1 = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{Span} < \begin{pmatrix} 1 \\ 0 \end{pmatrix} > \Rightarrow m_g(1) = 1 \neq m_a(1)$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Lemma: $e^{t(I+N)} = e^{tI} e^{tN}$ se I e N commutano $(IN = NI)$

Poiché nel nostro caso $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutano banalmente (I identità!)

$$e^{tA} = e^{t(I+N)} = e^{tI} e^{tN} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}(\mathbb{1} + tN) = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

N è nilpotente

$N^2 = 0$

Quindi l'integrale generale

$$I = e^{tA}\underline{c} = e^{t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

D. $F : \mathbb{R}^2 \longmapsto \mathbb{R}^2$

$$F(x,y) = (x+y,\ x^3 - y) \qquad F \in \mathcal{C}^1(\mathbb{R}^2)$$

$$dF(x,y) = \begin{pmatrix} 1 & 1 \\ 3x^2 & -1 \end{pmatrix}$$

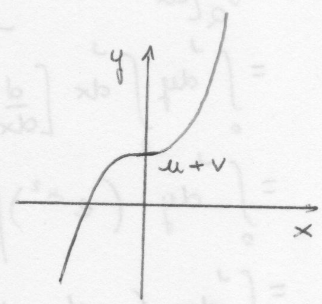
$$|dF(x,y)| = -1 - 3x^2 \neq 0 \quad \forall \binom{x}{y} \in \mathbb{R}^2 \implies \text{diffeomorfismo locale}$$

F è biunivoca?

$$\begin{cases} u = x + y \\ v = x^3 - y \end{cases}$$

$$\begin{cases} u + v = x + x^3 \\ y = u - x \end{cases}$$

$$\begin{cases} x^3 + x - (u+v) = 0 \\ y = u - x \end{cases}$$



$f(x) = x^3 + x - (u+v)$

$f'(x) = 3x^2 + 1 \implies f$ crescente

$f''(x) = 6x \implies f$ concava per $x < 0$

convessa per $x > 0$

flesso per $x = 0$

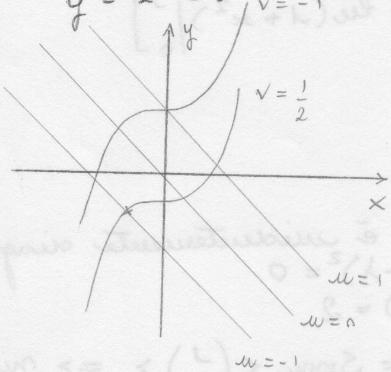
$f(x) = 0$ ammette una e una sola soluzione $\bar{x}$

$\implies \exists!\ \bar{y} = u - \bar{x}$

$\implies F^{-1}$ è ben definita $\implies$ diffeomorfismo globale

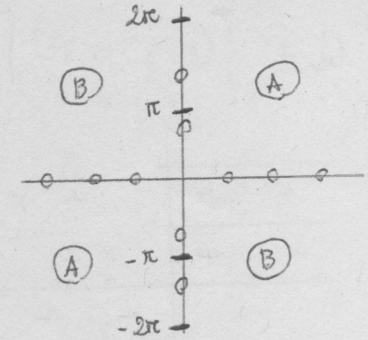
Intuitivamente: $y = -x + u$

$y = x^3 - v$



$\forall$ punto $\binom{x}{y} \in \mathbb{R}^2$ $\exists!$ retta del fascio che passa per $\binom{x}{y}$ e un'unica cubica con la stessa proprietà (rette e cubiche sono parametrizzate da (u,v)) e viceversa.

1. $f: \mathbb{R}^2 \longmapsto \mathbb{R}$



$$f(x,y) = \begin{cases} x^2y^2e^{1-(x^2+y^2)} & xy > 0 \qquad \text{(A)} \\ x\sin y & xy \leq 0 \qquad \text{(B)} \end{cases}$$

$xy \neq 0 \Rightarrow$ vale il Teorema del differenziale totale
$f \in C^\infty(\mathbb{R}^2 \setminus \{xy \neq 0\})$

$h(x,y) = x^2y^2e^{1-(x^2+y^2)}$

$\nabla h = \left[2xy^2e^{1-(x^2+y^2)}(1-x^2),\ 2yx^2e^{1-(x^2+y^2)}(1-y^2) \right]$

$g(x,y) = x\sin y$

$\nabla g = [\sin y,\ x\cos y]$

N.B. f continua sugli assi: basta fare il limite di $h(x,y)$ con $(x,y) \longmapsto \begin{matrix} (x,0) & x \neq 0 \\ (0,y) & y \neq 0 \end{matrix}$

asse $\vec{y} = \{x = 0,\ y \neq 0\}$ $\quad \partial_x f_A = 0 \quad \partial_y f_B = \sin y \neq 0$ per $y \neq k\pi$

$\Rightarrow f$ non è diff. su $\vec{y}$ ad ecc. $(0, k\pi)$

asse $\vec{x} = \{y = 0,\ x \neq 0\}$ $\quad \partial_y f_A = 0 \quad \partial_y f_B = x\cos y = x \neq 0$

$\Rightarrow f$ non è diff. su $\vec{x}$

$(0,0)$ $\quad f(0,0) = 0$

$$\lim_{(h,k)\mapsto(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = 0$$

$$\left| \frac{f(h,k)}{\sqrt{h^2+k^2}} \right| \leq \max\left\{ \frac{|h^2k^2e^{1-(h^2+k^2)}|}{\sqrt{h^2+k^2}},\ \frac{|x\sin y|}{\sqrt{h^2+k^2}} \right\}$$

$$\leq \max\left\{ \frac{e\,h^2k^2}{\sqrt{h^2+k^2}},\ \frac{|hk|}{\sqrt{h^2+k^2}} \right\}$$

$$\leq \max\left\{ \frac{e}{2}\sqrt{h^2+k^2}\,hk,\ \frac{1}{2}\sqrt{h^2+k^2} \right\} \leq \frac{e}{2}\sqrt{h^2+k^2} \xrightarrow[\binom{h}{k}\mapsto\binom{0}{0}]{} 0$$

$\Rightarrow f$ diff. le in $(0,0)$

$(0, n\pi)$ $\quad f(0, n\pi) = 0$

$$\lim_{(h,k)\mapsto(0,0)} \frac{f(h, n\pi+k)}{\sqrt{h^2+k^2}} = 0$$

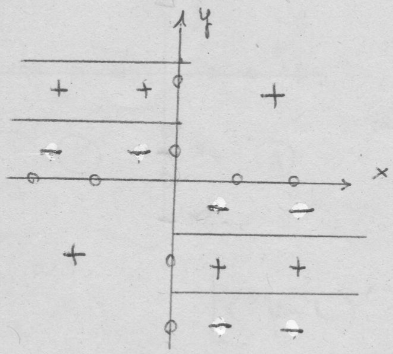
$$\left| \frac{f(h, n\pi+k)}{\sqrt{h^2+k^2}} \right| \leq \max\left\{ \frac{h^2(n\pi+k)^2e^{1-(h^2+(n\pi+k)^2)}}{\sqrt{h^2+k^2}},\ \frac{|h\sin(n\pi+k)|}{\sqrt{h^2+k^2}} \right\}$$

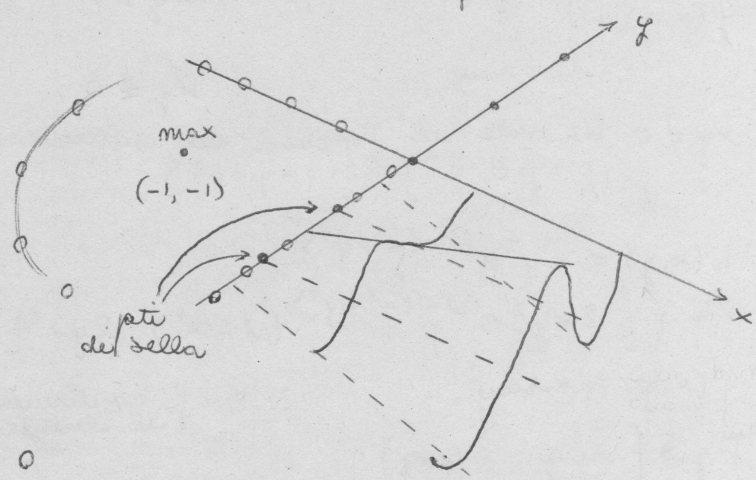
$$\leq \max\left\{ \underbrace{(n\pi+1)^2e^{1-n^2\pi^2}}_{C} \cdot \frac{h^2e^{-h^2-k^2}}{\sqrt{h^2+k^2}},\ \underbrace{\frac{|hk|}{\sqrt{h^2+k^2}}}_{\leq \frac{1}{2}\sqrt{h^2+k^2}} \right\} \xrightarrow[\binom{h}{k}\mapsto\binom{0}{0}]{} 0$$

in coord. polari

$$C\,\frac{\rho^{\cancel{2}}\cos^2\theta\,\cancel{e^{-\rho^2}}}{\cancel{\rho}} \leq C\rho\,\cancel{e^{-\rho^2}} \xrightarrow[\rho\mapsto 0]{} 0$$

$Q =: \{(x,y) \in \mathbb{R}^2 \mid |x| < 2, |y| < 2\}$ per simmetria studiamo solo due quadranti





$$\lim_{\binom{h}{k} \mapsto \binom{0}{0}} h^2 k^2 e^{1-(h^2+k^2)} = 0$$

in coord. polari $\rho^4 \cos^2\theta \sin^2\theta \, e^{1-\rho^2} \leq \rho^4 e^{1-\rho^2} \simeq \rho^4 e^{-\rho^2} \underset{\rho \mapsto 0}{\longmapsto} 0$

poiché $h(x,y) > 0$ nei quadranti,
mi aspetto almeno un max (generalizz. Teo di Rolle)

pti stazionari: $(\pm 1, \pm 1) \Rightarrow (-1,-1), (1,1)$ sono pti di max

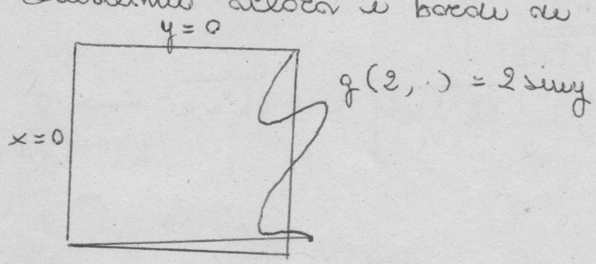
i pti $(0, k\pi)$ sono di sella (per lo studio del segno)

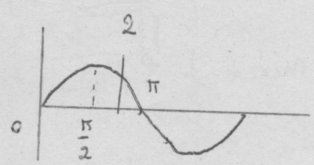
Per quanto riguarda $g(x,y)$

$g(\bar{x}, \cdot) = \bar{x} \sin y \Rightarrow$ ammette max per $x = \frac{\pi}{2} + 2k\pi$
min per $x = \frac{3}{2}\pi + 2k\pi$

$g(\cdot, \bar{y}) = x \sin \bar{y} \Rightarrow$ ammette max per $x = 2$
min per $x = 0$ } su Q

Studiamo allora i bordi di B





$g(\cdot, 2) = x \sin 2$

$\Rightarrow \min\big|_B = \left(-2, \frac{\pi}{2}\right)$ (Si consideri il segno di f)

$f\left(-2, \frac{\pi}{2}\right) = f\left(+2, \frac{3\pi}{2}\right) = -2$

$\min\big|_B = \left(2, \frac{3}{2}\pi\right)$

$\max\big|_A = (1,1) = (-1,-1)$ $f(1,1) = f(-1,-1) = \frac{1}{e}$

$\bar{Q}$ cmp, f continua $\Rightarrow$ ammette max e min

max $= (1,1) = (-1,-1)$ $\frac{1}{e}$
min $= \left(-2, \frac{\pi}{2}\right) = \left(2, \frac{3}{2}\pi\right)$ -2

su Q $\exists$ max in $(\pm 1, \pm 1)$ assume il valore $1/e$
e inf $= -2$

2. $f_n : \mathbb{R}^+ \longmapsto \mathbb{R} \qquad f_n(x) = \dfrac{x^n}{(x-1)^n + (x+1)^n}$

C.P. su $\mathbb{R}^+ \qquad f_n(x) = \dfrac{1}{\left(1-\frac{1}{x}\right)^n + \left(1+\frac{1}{x}\right)^n}$

$\left(1-\frac{1}{x}\right)^n \longmapsto 0$ per $x \in \left(\frac{1}{2}, +\infty\right)$

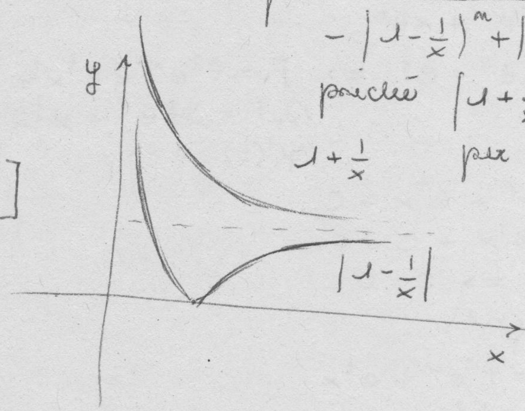
$\left(1+\frac{1}{x}\right)^n \longmapsto 0$ per $x \in \left(-\frac{1}{2}, +\infty\right)$

$\left(0, \frac{1}{2}\right) \qquad \dfrac{1}{\left(1-\frac{1}{x}\right)^n + \left(1+\frac{1}{x}\right)^n}$
- n pari: $\dfrac{1}{\left|1-\frac{1}{x}\right|^n + \left|1+\frac{1}{x}\right|^n} \longmapsto 0$ entrambe divergono
- n dispari: $\dfrac{1}{-\left|1-\frac{1}{x}\right|^n + \left|1+\frac{1}{x}\right|^n} \longmapsto 0$ perché $\left|1+\frac{1}{x}\right| > \left|1-\frac{1}{x}\right|$ per $x > 0$

$\left(\frac{1}{2}, +\infty\right) \quad f_n \sim \dfrac{1}{\left(1+\frac{1}{x}\right)^n} \longmapsto 0$

$\Rightarrow$ C.P. su $\mathbb{R}^+$

$\left[\, x = \frac{1}{2} \quad f_n = \dfrac{1}{(-1)^n + 3^n} \longmapsto 0 \,\right]$



C.U. $\quad x \longmapsto +\infty \quad f_n \longmapsto \frac{1}{2}$

$x = 0 \quad f_n = 0$

$x > 0 \quad f_n > 0$

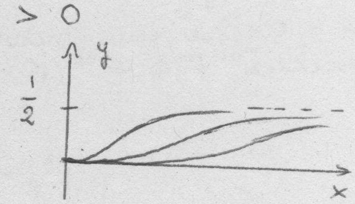
$x+1 > x-1$

$(x+1)^n > (x-1)^n$

$$f_n'(x) = \frac{-\frac{n}{x^2}\left[\left(1-\frac{1}{x}\right)^{n-1} - \left(1+\frac{1}{x}\right)^{n-1}\right]}{\left[\left(1-\frac{1}{x}\right)^n + \left(1+\frac{1}{x}\right)^n\right]^2} > 0$$

che la funz. potenza è crescente su $\mathbb{R}^+$

$\left(1+\frac{1}{x}\right) > \left|1-\frac{1}{x}\right| \geq \left(1-\frac{1}{x}\right) \quad \forall x \in \mathbb{R}^+$



$\sup_{x \in \mathbb{R}^+} |f_n(x)| = \frac{1}{2} \neq 0 \Rightarrow$ no C.U. su $\mathbb{R}^+$

$\sup_{x \in (0,M]} |f_n(x)| = f_n(M) \underset{n \mapsto +\infty}{\longmapsto} 0$ per C.P. $\Rightarrow$ C.U. su $(0, M]$

3. $f : \mathbb{R}^2 \mapsto \mathbb{R}$

$$f(t,x) = - \frac{2te^x + xe^t}{t^2e^x + e^t} = - \frac{P(t,x)}{Q(t,x)}$$

$x' = f(t,x) \qquad x(0) = 0$

$$\frac{dx}{dt} = - \frac{P(t,x)}{Q(t,x)}$$

Si cerca $F(t,x)$ t.c. $F(t,x) = 0$ definisce implicit. la sol. del (PC)

$$\varphi' = - \frac{\partial_x F(t,\varphi)}{\partial_y F(t,\varphi)}$$

Si consideri la forma diff. ω

$\omega = (t^2e^x + e^t)\,dx + (2te^x + xe^t)\,dt = Q(t,x)\,dx + P(t,x)\,dt$

$\partial_t Q = \partial_x P$; infatti $\partial_t Q = 2te^x + e^t = 2te^x + e^t = \partial_x P \Rightarrow \omega$ è chiusa su $\mathbb{R}^2$
$\Rightarrow \omega$ esatta

$\partial_t F = 2te^x + xe^t$

$\frac{\partial F}{\partial x} = t^2e^x + e^t \Rightarrow F = t^2e^x + e^tx + \psi(t)$

$\partial_t F = \cancel{2te^x} + \cancel{e^t x} + \psi'(t) = \cancel{2te^x} + \cancel{xe^t}$

$\psi'(t) = 0$

$F = t^2e^x + e^tx + C$

$t^2e^x + e^tx + C = 0$

$x(0) = 0 \Rightarrow C = 0$

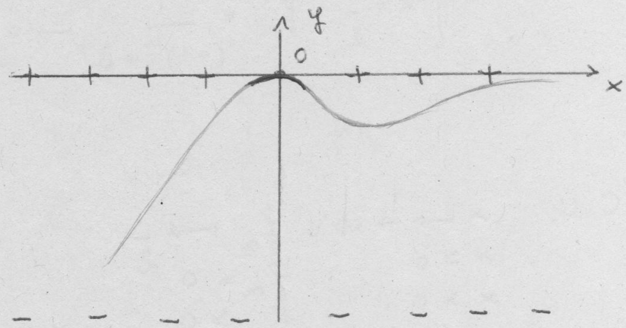
$F \in \mathcal{C}^\infty(\mathbb{R}^2)$

$F(t,x) = t^2e^x + e^tx$

$F(t,0) = t^2$

$\lim_{x \mapsto -\infty} F(t,x) = -\infty$

$\partial_x F = t^2e^x + e^t > e^t \quad \forall (t,x)$



$\Rightarrow$ è definita una (soluzione) funz. implicita da $\mathbb{R} \mapsto \mathbb{R}^-$ (non pos.)

$x = 0$ è un massimo per la funz.

poiché $F \in \mathcal{C}^\infty(\mathbb{R}^2) \Rightarrow \varphi \in \mathcal{C}^\infty(\mathbb{R}) \Rightarrow x = 0$ pto stazionario e max assoluto (non esistono altri $\bar{x}$ t.c. $\varphi(x) \geq 0$)

pti stazionari $\begin{cases} t^2e^x + e^tx = 0 \\ 2te^x + xe^t = 0 \end{cases}$

$(t^2 + 2t)e^{\cancel{x}} = 0$

$t = 0, \ t = 2$

Segno della derivata: $2te^x + xe^t > 0$

$2te^x + \cancel{xe^t} - t^2e^x - \cancel{e^tx} > 0$

$2t - t^2 > 0$

$0 < t < 2$

$\Rightarrow$ (ricordando che $\partial_x F > 0$)

$\varphi' > 0 \qquad t < 0 \vee t > 2$

$\varphi' < 0 \qquad 0 < t < 2$

$\Rightarrow t = 2$ pto di min.

limite all'infinito: per la monotonia di φ esistono

$$0 = \lim_{t \mapsto -\infty} t^2e^x + e^tx$$

$\varphi \mapsto \ell \quad \sim t^2 e^{\ell} + e^t \ell \mapsto +\infty$, assurdo

$\Rightarrow \varphi \mapsto -\infty \quad t \mapsto -\infty$

poiché per $t > 2$ la funzione è crescente e limitata da $\varphi = 0$ deve ammettere AS. ORIZZ.

Supponiamo $\ell \neq 0$

$$0 = \lim_{t \mapsto +\infty} t^2 e^x + e^t x$$

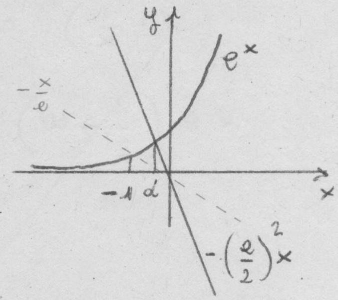
$$\varphi \mapsto \ell \neq 0 \qquad t^2 e^{\ell} + e^t \ell \mapsto \cancel{+}\infty \quad \begin{bmatrix} \ell \cdot e^t \mapsto -\infty \\ e^{\ell} \cdot t^2 \mapsto +\infty \end{bmatrix}$$

$\Rightarrow \varphi \mapsto 0 \quad t \mapsto +\infty$

Stima del minimo: $t = 2 \qquad 4e^x + e^2 x = 0$

$$e^x = -\left(\frac{e}{2}\right)^2 x$$

$$-1 < \varphi(2) < 0$$



Asintoto obliquo: $\displaystyle \varphi' = -\frac{2te^x + xe^t}{t^2 e^x + e^t}$ [Aggiungiamo a N e D 0]

$$= -\frac{2te^x + \cancel{xe^t} - t^2 e^x - \cancel{e^t x}}{\cancel{t^2 e^x} + e^t - \cancel{t^2 e^x} - e^t x}$$

$$= \frac{(t^2 - 2t) e^x}{e^t (1 - x)}$$

$$\begin{matrix} x \mapsto -\infty \\ t \mapsto -\infty \end{matrix} \quad \varphi' \sim \frac{t^2 - 2t}{e^t} \cdot \left[-\frac{e^x}{x}\right] = \frac{t^2 - 2t}{\cancel{e^t}} \cdot \frac{\cancel{e^t}}{t^2} \mapsto 1$$

$$t^2 e^x + e^t x = 0$$

$$-\frac{e^x}{x} = \frac{e^t}{t^2}$$

$$\lim_{t \mapsto -\infty} \varphi - t = \div\infty$$

$$t^2 e^x + e^t x = 0$$

$$e^{x-t} = -\frac{x}{t^2}$$

$$x - t = \ln\left(-\frac{x}{t^2}\right) \mapsto -\infty$$

$$\lim_{t \mapsto -\infty} -\frac{x}{t^2} = \lim_{t \mapsto -\infty} -\frac{x'}{2t} \sim -\frac{1}{2t} \mapsto 0$$

$\Rightarrow$ non ammette AS. OBLIQUO